International Research Training Group IGDK 1754

IGDK Students' Workshop

Symbol of the Hessian

Lucas Bonifacius

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1 Motivation

The term *symbol* originates from Fourier analysis. Since differentiation corresponds to multiplication in the Fourier space, i.e.

$$\hat{f}'(\xi) = 2\pi\xi\hat{f}(\xi),$$

we may identify the differential operator of order 1 by the symbol $2\pi\xi$.

Our aim is to find an easy approximation of the Hessian $j''(u)$ by characterizing the mapping

$$v \mapsto \langle j''(u)v, v\rangle,$$

where j denotes the reduced objective function.

In his PhD thesis, Moritz develops a structured approach to obtain an approximation of the shape Hessian. Using the approximation as a preconditioner for CG methods and in Newton-type methods he obtains significant performance improvement in terms of fewer CG respectively Newton iterations.

It seems to be unclear if this approach might also been used in other types of optimization problems. Thus, the aim of the project is to apply his method to other problem classes to obtain - in a first step - an approximation of the Hessian. Thereafter, one could try to investigate its usefulness in applications.

2 Method

The method developed by Moritz can be summarized to the following steps:

(1) Localize problem

(2) Choose perturbation v

(3) Characterize the mappings $v \mapsto S'(q)v$ and $v \mapsto S''(q)v^2$

(4) Express $\langle j''(q)v, v\rangle$ in terms of v, v', etc.

3 Example: Semilinear elliptic boundary control

As an example, I considered the following semilinear elliptic Neumann boundary control problem, with state u and control q.

$$\begin{aligned} &\text{Minimize } j(q) := \frac{1}{2}\int_\Gamma (u(q) - u_d)^2, \\ &-\Delta u + u^3 - 1 = 0, && \text{in } \Omega, \\ &\qquad\qquad \partial_n u = q, && \text{on } \partial\Omega. \end{aligned} \tag{P}$$

Let $S(q) = u$ denote the control-to-state mapping. Then the linearized state equation $S'(q)v = z$ is given by

$$\begin{aligned} -\Delta z + 3u^2 z &= 0, && \text{in } \Omega, \\ \partial_n z &= v, && \text{on } \partial\Omega. \end{aligned}$$

The second linearized state equation $S''(q)v^2 = \tilde{z}$ is

$$\begin{aligned} -\Delta \tilde{z} + 3u^2 \tilde{z} &= -6uz^2, && \text{in } \Omega, \\ \partial_n \tilde{z} &= 0, && \text{on } \partial\Omega. \end{aligned}$$

Following the approach of Moritz we consider an auxillary localized optimal control problem by zooming in on some point of the control boundary, so that it appears flat, i.e. $\Gamma = \{x \in \mathbb{R} \colon x_2 = 0\}$. See the following sketch:

We now choose the perturbation

$$v(x) = \mathrm{e}^{\mathrm{i}\rho x_1}$$

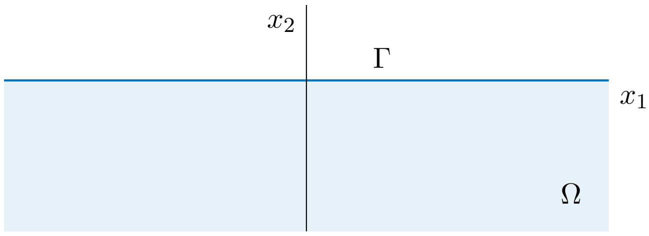

Figure 1: Localized problem

one may obtain the expressions (?)

$$\begin{aligned} \langle j_{uu}(S(0),0)S'(0)v, S'(0)v\rangle &= \int_\Gamma \beta^{-2}v^2, \\ \langle j_u(S(0),0), S''(0)(v,v)\rangle &= \int_\Gamma (u_d - S(0))\left(\frac{2}{3}\beta^{-1} - \frac{4}{3}\gamma^{-1}\right)v^2, \end{aligned}$$

where $\beta = \sqrt{\rho^2+3}$ and $\gamma = \sqrt{4\rho^2+3}$. Taylor expansion leads to the approximations

$$\begin{aligned} \langle j_{uu}(S(0),0)S'(0)v, S'(0)v\rangle &\approx \frac{1}{3}\int_\Gamma v^2 + \frac{1}{36}\int_\Gamma (v')^2, \\ \langle j_u(S(0),0), S''(0)(v,v)\rangle &\approx -\frac{1}{\sqrt{3}}\int_\Gamma (u_d - S(0))\left(\frac{2}{3}v^2 + \frac{1}{12}(v')^2\right). \end{aligned}$$

However, the approximation is not positive definite and numerical examples show (?) that these are no appropriate approximations.

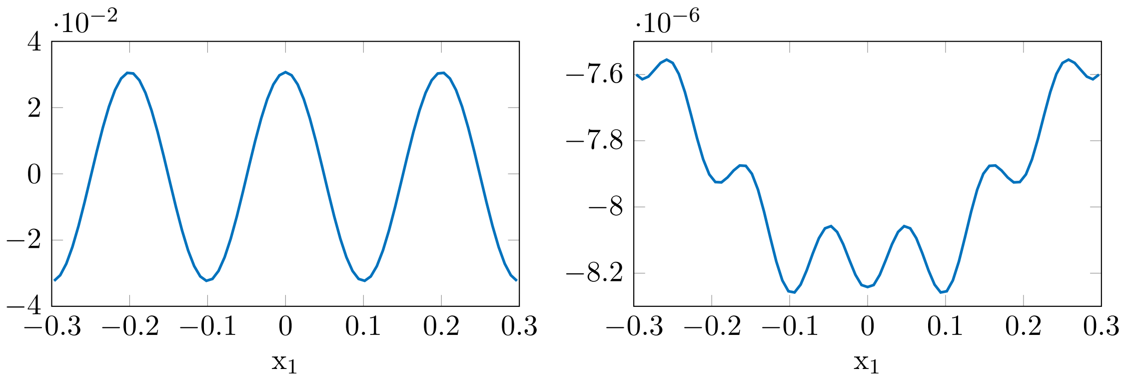

Figure 2: Linearized and second linearized state for $v = \cos(10\pi x_1)$, boundary view at $x_2 = 0$

4 Applications

There might be numerous applications that might be figured out in the workshop, for example

- Newton-type methods
- Globalized SQP methods
- Preconditioning in CG methods

References

[1] K. Eppler et al. “Preconditioning the pressure tracking in fluid dynamics by shape Hessian information”. In: *J. Optim. Theory Appl.* 141.3 (2009), pp. 513–531. issn: 0022-3239. doi: 10.1007/s10957-008-9507-y.

[2] S. Schmidt. “Efficient Large Scale Aerodynamic Design Based on Shape Calculus”. PhD thesis. University of Trier, 2010.