# 1 Max Plus Basis Method for Solution of Hamilton-Jacobi-Bellman PDEs

## 1.1 Introduction

There are two approaches in nonlinear control: the maximum principle approach and the dynamic programming approach. The dynamic programming yields a necessary and a sufficient optimality condition for every initial state. On the other hand, the maximum principle yields only a necessary optimality condition for one initial state. However, from the computational point of view, the maximum principle has the large advantage that it involves only ordinary differential equations, whereas the dynamic programming leads to a fully nonlinear partial differential equation, the Hamilton-Jacobi-Bellman (HJB) PDE.

Solving the HJB PDE is in general a very difficult task. The most common methods fall into the category of grid-based methods, e.g. finite difference or finite element methods. The methods of interest, namely the max-plus basis method, also fall into this category. The largest problem with these methods is that they are subject to the "curse of dimensionality", i.e. the computational time grows exponentially with the state space dimension. The max-plus basis methods have the advantage over the other grid-based methods in that they need considerably smaller amount of grid points and a larger time step.

## 1.2 Max-plus Basis Methods

The max-plus algebra is a commutative semifield over $\mathbb{R}^{-} = \mathbb{R} \cup \{-\infty\}$. The addition and multiplication operations are:

$$a \oplus b = \max\{a, b\},$$
$$a \otimes b = a + b.$$

The additive identity is $-\infty$, and the multiplicative identity is 0. Commutative, associative and distributive properties hold.

We say $X$ is a max-plus vector space (with the zero element denoted by $\phi_0 \in X$) if

$$
\begin{aligned}
& a \otimes \phi \in X \quad \forall a \in \mathbb{R}^-,\ \forall \phi \in X \\
& \phi \oplus \psi = \psi \oplus \phi \in X \quad \forall \psi,\ \phi \in X, \\
& (a \otimes b) \otimes \phi = a \otimes (b \otimes \phi), \quad \forall a,\ b \in \mathbb{R}^-,\ \forall \phi \in X \\
& (a \oplus b) \otimes \phi = (a \otimes \phi) \oplus (b \otimes \phi), \quad \forall a,\ b \in \mathbb{R}^-,\ \forall \phi \in X \\
& a \otimes (\phi \oplus \psi) = (a \otimes \phi) \oplus (a \otimes \psi), \quad \forall a \in \mathbb{R}^-,\ \forall \phi,\ \psi \in X \\
& \phi \oplus \phi_0 = \phi, \quad a \otimes \phi_0 = \phi_0, \quad -\infty \otimes \phi = \phi_0, \quad 0 \otimes \phi = \phi, \quad \forall a \in \mathbb{R}^-,\ \forall \phi \in X.
\end{aligned}
$$

The max-plus basis method relies on the fact that the control problem, which is fully nonlinear in the standard algebra, becomes linear in max-plus algebra. The solution is found as a max-plus basis expansion over a suitable max-plus vector space (specifically, the space of semiconvex functions).

### 1.2.1 Problems of Interest

Consider the infinite linear quadratic regulator problem

$$
\begin{aligned}
\dot{x} &= Ax + Bu, \\
x(t_0) &= x_0,
\end{aligned}
$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$; and the cost functional is

$$
J(u) = \int_0^\infty (x^T Q x + u^T R u) dt,
$$

where $Q$ is symmetric positive semidefinite, and $R$ is symmetric positive definite. (All matrices $A$, $B$, $Q$ and $R$ are constant matrices of the appropriate dimensions.) The HJB PDE for this problem is analitically solvable (via the algebraic Riccati equation).

The idea is to implement the max-plus method for this problem in the cases $n = m = 2$, and $n = m = 3$. Having an actual solution for the comparison should be helpful.

# References

[1] McEneaney, W. M. (2006). Max-plus Methods for Nonlinear Control and Estimation

*Springer, 2006.*